P

NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

Faculty of Health, Natural Resources and Applied Sciences

School of Natural and Applied Sciences

Department of Mathematics, Statistics and Actuarial Science

13 Jackson Kaujeua Street
Private Bag 13388
Windhoek
NAMIBIA

T: +264 61 207 2913
E: msas@nust.na
W: www.nust.na

QUALIFICATION : **BACHELOR of SCIENCE IN APPLIED MATHEMATICS AND STATISTICS**	
QUALIFICATION CODE: **07BSAM**	LEVEL: **6**
COURSE: **STATISTICAL INFERENCE 2**	COURSE CODE: **SIN601S**
DATE: **NOVEMBER 2024**	SESSION: **1**
DURATION: **3 HOURS**	MARKS: **100**

FIRST OPPORTUNITY EAMINATION QUESTION PAPER

EXAMINER:	Dr J MWANYEKANGE
MODERATOR:	Dr D. B. GEMECHU

INSTRUCTIONS:

1. Answer all questions on the separate answer sheet.
2. Write your answers neatly and legibly.
3. Do not use the left side margin of the exam paper. This must be allowed for the examiner.
4. No books, notes and other additional aids are allowed.
5. Mark all answers clearly with their respective question numbers.

PERMISSIBLE MATERIALS:

1. Non-Programmable Calculator without cover

This paper consists of 4 pages including this front page

Question 1 [15 Marks]

1.1. Let $Y_1 < Y_2 < \cdots < Y_5$ be the ordered statistics of 5 independently and identically distributed continuous random variables $X_1, X_2, \ldots, X_5$ with pdf f given by

$$f_X(x) = \begin{cases} 6x^2, & for\ 0 < x < 1 \\ 0, & Otherwise \end{cases}$$

Then,

1.1.1. Show that the cumulative density function of X is $F_X(x) = 2x^3$ [3]

1.1.2. Find the pdf of the minimum order statistics [3]

1.1.3. Find the pdf of the maximum order statistics [3]

1.1.4. Find the joint pdf of the 2nd and 5th order statistics [6]

$Hint: f_{Y_i,Y_j}(y_i, y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F_X(y_i)]^{i-1}[F_X(y_j) - F_X(y_i)]^{j-i-1}[1 - F_X(y_j)]^{n-j} f_X(y_i) f_X(y_j)$

Question 2 [13 marks]

2.1 Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed random variable with normal distribution having $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$.

Show, using the moment generating function, that $Y = \sum_{i=1}^{n} X_i$ has a normal distribution with $\mu_Y = n\mu$ and $\sigma_Y^2 = n\sigma^2$. [8]

$\left(\boldsymbol{Hint}: If\ X \sim N(\mu, \sigma^2), then\ M_{X_i}(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)$

2.2 If $X \sim \chi_m^2$ and $Y \sim \chi_n^2$ such that X and Y are independent, what is the distribution of $X + Y$? [5]

Question 3 [8 marks]

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of observations from a Bernoulli distribution

Show that $T = \frac{y(y-1)}{n(n-1)}$ is an unbiased estimator of θ^2 where $y = \sum_{i=1}^{n} x_i$. (Hint: $E(y) = n\theta$ and $Var(y) = n\theta(1 - \theta)$ [8]

Question 4 [44 marks]

4.1 If $X \sim \Gamma(\alpha, \theta)$ a random sample of n observations $X_1, X_2, \dots, X_n$ is selected from a population X_i for $i = 1,2 \dots, n$ posses a gamma probability density function with parameters α and θ. Use the method of moment to estimate α and θ. **Hint**: $E(X) = \alpha\theta$, $Var(X) = \alpha\theta^2$ and $M_x(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha}$ [8]

4.2 Let $X_1, X_2, \dots, X_n$ be a random variable from a Bernoulli distribution with pdf:

$$f(X;p) = p^X(1-p)^{1-X}, \quad X = 0,1.$$

4.2.1 Using the m.g.f of X, show that the mean and variance of X_i are p and $\text{p}(1-p)$, respectively ($\boldsymbol{Hint}$: $M_{X_i}(t) = pe^t + q$) [6]

4.2.2 Find the maximum likelihood estimate of p. [6]

4.2.3 Let $T_n = \sum_{i=1}^{n} X_i$, show that T_n is sufficient for p.

$\left(\boldsymbol{Hint}: f(T_n) = \binom{n}{T_n} p^{T_n}(1-p)^{n-T_n}\right)$ [8]

4.2.4 Show that the $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a minimum variance unbiased estimator (MVUE) of p. $\left(\boldsymbol{Hint}: CRB = \frac{1}{nE\left(\frac{\partial}{\partial p}\log f(x;p)\right)^2}\right)$ [16]

Question 5 [20 marks]

5.1 Suppose that the prior distribution of θ follow a Gamma distribution with shape $\alpha = 2$ and rate β,

$$h(\theta) = \beta^2\theta e^{-\beta\theta} \quad for\ \theta > 0$$

Given θ, X is uniform over the interval $(0, \theta)$ with pdf given by

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta \\ 0, & Otherwise \end{cases}$$

What is the posterior distribution of θ. $\left(\boldsymbol{Hint}: \varphi(\theta|x) = \frac{f(x|\theta)h(\theta)}{\int_x^{\infty} f(x|\theta)h(\theta)d\theta}\right)$ [8]

5.2 Let $X_1, \dots, X_n$ be random samples from the binomial distribution:

$$f(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad for\ x = 1,2, \dots, n$$

The prior distribution of θ is a beta distribution with the parameters α and β,

$$h(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,.\,\Gamma(\beta)}\,\theta^{\alpha-1}\,(1-\theta)^{\beta-1}, & for\ 0 < \theta < 1 \\ 0, & Otherwise \end{cases}$$

Show that posterior distribution of θ given $X = x$ is a beta distribution with parameter $x + \alpha$ and $n - x + \beta$. $\left(\boldsymbol{Hint}\text{: } \int_0^1 \theta^{x+\alpha-1}\,(1-\theta)^{n+\beta-1} d\theta = \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(n+\alpha+\beta)}\right)$ [12]

==========================**TOTAL MARKS 100**==========================