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QUALIFICATION : **BACHELOR OF SCIENCE IN APPLIED MATHEMATICS HONOURS**	
QUALIFICATION CODE: **08BSHM**	LEVEL: **8**
COURSE: **FUNCTIONAL ANALYSIS**	COURSE CODE: **FAN802S**
DATE: **NOVEMBER 2023**	SESSION: **1**
DURATION: **3 HOURS**	MARKS: **100**

FIRST OPPORTUNITY: EXAMINATION QUESTION PAPER

EXAMINER: *Dr S.N. NEOSSI-NGUETCHUE*

MODERATOR: *Prof F. MASSAMBA*

INSTRUCTIONS:

1. Answer all questions on the separate answer sheet.
2. Please write neatly and legibly.
3. Do not use the left side margin of the exam paper. This must be allowed for the examiner.
4. No books, notes and other additional aids are allowed.
5. Show clearly all the steps used in the calculations.
6. Mark all answers clearly with their respective question numbers.

PERMISSIBLE MATERIALS:

1. Non-Programmable Calculator

ATTACHEMENTS

None

This paper consists of 2 pages including this front page

Problem 1: [27 Marks]

1-1. Let $X \neq \emptyset$. Give the definition of the following concepts:

1-1-1. A σ-algebra on X and a σ-algebra generated by a family $\mathcal{C}$ of subsets of X. [3+2]

1-1-2. A Borel σ-algebra on X. [3]

1-1-3. A measurable space on X. [1]

1-1-4. A measure on X. [3]

1-1-5. A measure space on X. [1]

1-2. Let $E \subset \mathbb{R}$ a non-empty set. Show that $\mathcal{F} = \{\emptyset, E, E^c, \mathbb{R}\}$ is the σ-algebra of subsets of $\mathbb{R}$ generated by $\{E\}$. [9]

1-3. Let $X = \{1, 2, 3, 4\}$ and consider $\mathcal{C} = \{\{1\}, \{2, 3\}\} \subset \mathcal{P}(X)$. Determine $\sigma(\mathcal{C})$ the σ-algebra generated by $\mathcal{C}$. [5]

Problem 2: [35 Marks]

Let $(X, \|\cdot\|)$ be a normed space.

2-1. Assume that X is a Banach space.
Show that any absolutely summable series is summable. [6]

2-2. Now we assume that X is a normed space in which any absolutely summable series is summable.

2-2-1. Let $\{x_n\}$ be a Cauchy sequence in X. Show that if $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, $\{x_n\}$ converges to the same limit. [6]

2-2-2. Show that we can construct a subsequence $\{x_{\varphi(n)}\}$ such that [6]

$$\forall k \in \mathbb{N}, \|x_{\varphi(k)} - x_{\varphi(k-1)}\| \leq \frac{1}{2^{k-1}}$$

and show that [6]

$$x_{\varphi(n)} = \sum_{k=1}^{n} (x_{\varphi(k)} - x_{\varphi(k-1)}) + x_{\varphi(0)}, \text{ for any } n \geq 1.$$

2-2-3. Deduce from question **2-2-2** that the sequence $\{x_{\varphi(n)}\}$ converges. [6]

2-2-4. Conclude that $\{x_n\}$ converges and therefore X is a Banach space. [3]

2-3. What is the general rule that you can establish from the main results obtained above. [2]

Problem 3: [38 Marks]

3-1. Consider $(\mathbf{X}, \|\cdot\|_{\infty,1})$, where $\mathbf{X} = \mathcal{C}^1[0,1]$ and $\|f\|_{\infty,1} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$ and also consider $(\mathbf{Y}, \|\cdot\|_\infty)$, where $\mathbf{Y} = \mathcal{C}[0,1]$.

3-1-1. Show that $T = \dfrac{d}{dx} \colon \mathbf{X} \to \mathbf{Y}$ is a bounded linear operator. [7]

3-1-2. Show that $T = \dfrac{d}{dx} \colon D(T) \subsetneq \mathbf{Y} \to \mathbf{Y}$ is an unbounded linear operator, where $D(T) = \mathcal{C}^1[0,1]$. [10]
(Hint: use $u_n(x) = \sin(n\pi x)$).

3-2. We recall that ℓ^2 or ℓ_2 sometimes denoted $\ell^2(\mathbb{N}_0)$ is the space of sequences defined by

$$\ell^2 = \left\{ x = (x_n)_{n\geq 0} \colon \sum_{n=0}^{\infty} |x_n|^2 < \infty \right\},\ \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \text{ and } \|x\|_{\ell^2} = \left(\sum_{n=0}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}.$$

Show that the following operators are linear and continuous and compute their norms.

3-2-1. $T_1 \colon \ell^2 \to \ell^2 \ :\ T_1\left((x_n)_{n\geq 0}\right) = (x_{n+1})_{n\geq 0}$. [9]

3-2-2. $T_2 \colon L^2([0,1]) \to \mathbb{C} : T_2(f) = \int_0^1 x^2 f(x)dx$, where: [12]

$L^2([0,1]) = \left\{ f : [0,1] \to \mathbb{R} \colon \int_0^1 |f(x)|^2 dx < \infty \right\}$ and $\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}}$.

God bless you !!!