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QUALIFICATION : **BACHELOR OF SCIENCE IN APPLIED MATHEMATICS HONOURS**	
QUALIFICATION CODE: **08BSHM**	LEVEL: **8**
COURSE: **FUNCTION ANALYSIS**	COURSE CODE: **FAN802S**
DATE: **NOVEMBER 2024**	SESSION: **1**
DURATION: **3 HOURS**	MARKS: **94**

FIRST OPPORTUNITY: QUESTION PAPER

EXAMINER: ***Dr SN NEOSSI-NGUETCHUE***

MODERATOR: ***Prof F. MASSAMBA***

INSTRUCTIONS:

1. Answer ALL the questions in the booklet provided.
2. Show clearly all the steps used in proofs and obtaining results.
3. All written work must be done in blue or black ink and sketches must be done in pencil.

PERMISSIBLE MATERIALS:

1. Non-Programmable Calculator

ATTACHEMENTS

None

This paper consists of 2 pages including this front page

Problem 1: [34 Marks]

1-1. Let $X \neq \emptyset$. Give the definition of the following concepts:
1-1-1. A σ-algebra on X and a σ-algebra generated by a family $\mathcal{C}$ of subsets of X. [2+2]
1-1-2. A Borel σ-algebra on X. [2]
1-1-3. A measurable space on X. [1]
1-1-4. A measure on X. [2]
1-1-5. A measure space on X. [1]

1-2. Let E a non-empty set and $A \in \mathcal{P}(E)$. Determine the σ-algebra generated by $\mathcal{C} = \{A\}$. [6]

1-3. Let $\mathcal{E}$ be a σ-algebra on X, and $X_0 \subset X$.
1-3-1. Show that $\mathcal{E}_0 = \{A \cap X_0 | A \in \mathcal{E}\}$ is a σ-algebra on X_0. [7]
1-3-2. Show that $\sigma(\mathcal{E}) = \mathcal{E}$. [5]

1-4. Let $\mathcal{K}, \mathcal{K}' \subset \mathcal{P}(X)$. Show that, if $\mathcal{K} \subset \mathcal{K}' \subset \sigma(\mathcal{K})$, then $\sigma(\mathcal{K}') = \sigma(\mathcal{K})$. [6]

Problem 2: [31 Marks]

We recall that ℓ^2 or ℓ_2 sometimes denoted $\ell^2(\mathbb{N}_0)$ is the space of sequences defined by

$$\ell^2 = \left\{ x = (x_n)_{n\geq 0} \colon \sum_{n=0}^{\infty} |x_n|^2 < \infty \right\}, \ \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \text{ and } \|x\|_{\ell^2} = \left(\sum_{n=0}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}.$$

2-1. We assume that $H = \ell^2$ is a complete space for the norm associated with $(\cdot,\cdot)_H$. Show that $H = \ell^2$ is a Hilbert space with respect to [10]

$$(x, y)_H = \sum_{n=1}^{\infty} x_n \bar{y}_n$$

2-2. Show that the following operators are linear and continuous and compute their norms.
2-2-1. $T_1 \colon \ell^2 \to \ell^2 \ : \ T_1\left((x_n)_{n\geq 0}\right) = (x_{n+1})_{n\geq 0}$. [9]
2-2-2. $T_2 \colon L^2([0,1]) \to \mathbb{C} : T_2\left(f\right) = \int_0^1 x^2 f(x)dx$, where: [12]
$L^2([0,1]) = \left\{ f : [0,1] \to \mathbb{R} \colon \int_0^1 |f(x)|^2 dx < \infty \right\}$ and $\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}}$.

Problem 3: [29 Marks]

3-1. State the Monotone Convergence Theorem (MCT) and the Dominated Convergent Theorem (DCT) respectively. [6]

3-2. Show that the function $f \colon (0, \infty) \to \mathbb{R}, f(x) := \dfrac{\sin x}{e^x - 1}, \forall\, x > 0$, is Lebesgue integrable on $[0, \infty]$. [6]

3-3. Show that for any $x > 0$, we have $f(x) = \displaystyle\sum_{n=1}^{\infty} e^{-nx} \sin x$. [5]

3-4. Deduce that $\displaystyle\int_0^{\infty} \frac{\sin x}{e^x - 1} = \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$. [12]

God bless you !!!