NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH, APPLIED SCIENCES AND NATURAL RESOURCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: Bachelor of science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 07BSAM	**LEVEL:** 6
COURSE CODE: SIN601S	**COURSE NAME**: STATISTICAL INFERENCE 2
SESSION: NOVEMBER 2022	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

FIRST OPPORTUNITY EXAMINATION QUESTION PAPER	
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INSTRUCTIONS

1. There are 5 questions, answer ALL the questions by showing all the necessary steps.
2. Write clearly and neatly.
3. Number the answers clearly.
4. Round your answers to at least four decimal places, if applicable.

PERMISSIBLE MATERIALS

1. Nonprogrammable scientific calculator

THIS QUESTION PAPER CONSISTS OF 3 PAGES (Including this front page)

Question 1 [29 Marks]

1.1. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistic of n independently and identically distributed continuous random variables $X_1, X_2, \ldots, X_n$ with probability density function f and cumulative distribution function F. Then, the cumulative distribution function of r^{th} order statistics, $F_{Y_r}(y)$ is given by

$$F_{Y_r}(y) = \sum_{k=r}^{n} \binom{n}{k} (F_X(y))^k (1 - F_X(y))^{n-k}$$

Use this result to show that the marginal distribution of the r^{th} order statistic is given by

$$f_{Y_r}(y) = \frac{n!}{(n-r)!(r-1)!} [F_X(y)]^{r-1} [1 - F_X(y)]^{n-r} f_X(y)$$ [10]

1.2. Suppose the random variables $X_1, X_2, \ldots, X_n$ are independently and identically distributed exponentially with the parameter θ, that is

$$f(x) = \begin{cases} \theta e^{-\theta x}, x > 0 \\ 0, \quad elsewhere \end{cases}$$

Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics for $X_1, X_2, \ldots, X_n$. Then,

1.2.1. Show that the cumulative density function of X is, $F_X(x) = 1 - e^{-\theta x}$ [3]
1.2.2. Find the probability density function of the minimum order statistic Y_1 [4]
1.2.3. Which density function does the p.d.f of Y_1 belongs to? [1]
1.2.4. Find the joint p.d.f. of $Y_1, Y_2, \ldots, Y_n$ [4]
1.2.5. If $n = 5$ and $\theta = 0.5$, then find
1.2.5.1. the probability that the sample maximum is greater than 2. [4]
1.2.5.2. the probability density function of the median. [3]

Question 2 [12 Marks]

2.1. Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed random variable with normal distribution having $E(X_i) = \mu$ and $V(X_i) = \sigma^2$. Then show, using the moment generating function, that $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ has a standard normal distribution. (**Hint:** If $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, *then* $M_{\bar{X}}(t) = e^{\mu t + \frac{\sigma^2 t^2}{2n}}$). [9]

2.2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Then find the expected value of $S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$.
Hint: $(n-1)\frac{S^2}{\sigma^2} \sim \chi^2(n-1)$ with mean $(n-1)$ and variance $2(n-1)$ [3]

Question 3 [23 Marks]

3.1. A random sample of n observations $X_1, X_2, \dots, X_n$ is selected for a population X_i, for $i = 1, 2, \dots, n$ which possesses a gamma probability density function with parameters α and θ. Use the method of moment to estimate α and θ.

(Hint: If $X \sim \text{Gamma}(\alpha, \theta)$, then the $M_X(t) = (1 - \theta t)^{-\alpha}$) [10]

3.2. Let $X_1, X_2, \dots, X_n$ denote a random sample from a distribution with density function

$$f_X(x|\theta) = \begin{cases} (1-\theta)x^{-\theta} & for\ 0 < x < 1 \\ 0 & otherwise. \end{cases}$$

Find maximum likelihood estimators of θ. [7]

3.3. Observations $Y_1, \dots, Y_n$ are assumed to come from a model with $E(Y_i) = 2 + \theta\, x_i$ where θ is an unknown parameter and $x_1, x_2, \dots, x_n$ are given constants. What is the least square estimate of the parameter θ? [6]

Question 4 [26 Marks]

4.1. Let $X_1, X_2, \dots, X_n$ denote a random sample a Rayleigh distribution with parameter θ.

$$f_X(x|\theta) = \begin{cases} 2\theta x e^{-\theta x^2} & for\ x > 0 \text{ and } \theta > 0 \\ 0 & otherwise \end{cases}$$

Show that $\sum_{i=1}^{n} x_i^2$ is sufficient for θ [5]

4.2. Suppose a random sample $X_1, X_2, \dots X_n$ is selected from a normally distributed population with unknown mean μ and variance σ^2.

4.2.1. Show that $\bar{X}$ is a minimum variance unbiased estimator (MVUE) of μ. [15]

4.2.2. Derive the $100(1-\alpha)\%$ CI for μ using the pivotal quantity method. [6]

Question 5 [10 Marks]

5. Suppose one observation was taken of a random variable X which yielded the value 2. The density function for X is

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & for\ 0 < x < \theta \\ 0 & otherwise \end{cases}$$

and the prior distribution of θ is

$$h(\theta) = \begin{cases} 2\theta^{-2} & for\ 1 < \theta < \infty \\ 0 & otherwise \end{cases}$$

5.1. Find the posterior distribution of θ. [7]

5.2. If the squared error loss function is used, find the Bayes' estimate of θ. [3]

=== END OF PAPER===

TOTAL MARKS: 100