NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

Faculty of Health, Natural Resources and Applied Sciences

School of Natural and Applied Sciences

Department of Mathematics, Statistics and Actuarial Science

13 Jackson Kaujeua Street
Private Bag 13388
Windhoek
NAMIBIA

T: +264 61 207 2913
E: msas@nust.na
W: www.nust.na

QUALIFICATION : **BACHELOR OF SCIENCE IN APPLIED MATHEMATICS HONOURS**	
QUALIFICATION CODE: **08BSHM**	LEVEL: **8**
COURSE: **FUNCTIONAL ANALYSIS**	COURSE CODE: **FAN802S**
DATE: **JANUARY 2024**	SESSION: **1**
DURATION: **3 HOURS**	MARKS: **100**

SECOND OPPORTUNITY / SUPPLEMENTARY: EXAMINATION QUESTION PAPER

EXAMINER: *Dr S.N. NEOSSI-NGUETCHUE*

MODERATOR: *Prof F. MASSAMBA*

INSTRUCTIONS:

1. Answer all questions on the separate answer sheet.
2. Please write neatly and legibly.
3. Do not use the left side margin of the exam paper. This must be allowed for the examiner.
4. No books, notes and other additional aids are allowed.
5. Show clearly all the steps used in the calculations.
6. Mark all answers clearly with their respective question numbers.

PERMISSIBLE MATERIALS:

1. Non-Programmable Calculator

ATTACHEMENTS

None

This paper consists of 2 pages including this front page

Problem 1: [45 Marks]

1-1. Let $f\colon \mathbb{R} \to \mathbb{R}$ such that $x \mapsto \begin{cases} 0, & \text{if } x \in \mathbb{Q}, \\ 1, & \text{if } x \notin \mathbb{Q}. \end{cases}$ Show that f is Borel-measurable. [10]

(Hint: for any $a \in \mathbb{R}$, consider $E = \{x \in \mathbb{R}\colon f(x) < a\}$ and show that $f^{-1}(E) \in \mathcal{B}(\mathbb{R})$)

1-2. Let $(\mathbf{X}, \mathcal{F})$ be a measurable space. Prove that if $A_n \in \mathcal{F}, n \in \mathbb{N}$, then $\displaystyle\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$. [5]

1-3. Let Ω be a non-empty set and $\mathcal{F}_\alpha \subset \mathcal{P}(\Omega), \alpha \in I$ an arbitrary collection of σ-algebras on Ω. State the definition of a σ-algebra and prove that [4+6=10]

$$\mathcal{F} := \bigcap_{\alpha \in I} \mathcal{F}_\alpha \quad \text{is a } \sigma\text{-algebra.}$$

1-4. Let $(\mathbf{X}, \mathcal{A}, \mu)$ be a measure space.

(i) What does it mean that $(\mathbf{X}, \mathcal{A}, \mu)$ be a measure space? [3]

(ii) Show that for any $A, B \in \mathcal{A}$, we have the equality: $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$. [7]

(Hint: Consider two cases: (i) $\mu(A) = \infty$ or $\mu(B) = \infty$; (ii) $\mu(A), \mu(B) < \infty$ and then express $A, B, A \cup B$ in terms of $A \setminus B, B \setminus A, A \cap B$ where necessary.)

1-5. Show that the following Dirichlet function is Lebesgue integrable but not Riemann integrable [10]

$$\chi := \mathbb{1}_{\mathbb{Q}\cap[0,1]}\colon [0,1] \to \mathbb{R}$$
$$x \mapsto \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

Problem 2: [20 Marks]

2-1. Define what is a compact set in a topological space. [3]

2-2. Show that $(0, 1]$ is not a compact set for usual topology of $\mathbb{R}$. [9]

2-3. Let E be a Hausdorff topological space and $\{a_n\}_{n\in\mathbb{N}}$ a sequence of elements of E converging to a. Show that $K = \{a_n | n \in \mathbb{N}\} \cup \{n\}$ is compact in E. [8]

Problem 3: [35 Marks]

3-1. Use the convexity of $x \mapsto e^x$ to prove the Arithmetic-Geometric Mean inequality: [5]

$$\forall x, y > 0, \text{ and } 0 < \lambda < 1, \text{ we have: } x^\lambda y^{1-\lambda} \leq \lambda x + (1 - \lambda)y.$$

3-2. Use the inequality in question **2-1** to prove **Young's inequality**: [6]

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \ \forall \alpha, \beta > 0, \text{ where } p, q \in (1, \infty)\colon \frac{1}{p} + \frac{1}{q} = 1.$$

3-3. Use the result in question **3-2** to prove Hölder's inequality: [7]

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}, \forall\, \mathbf{x} = (x_i),\ \mathbf{y} = (y_i) \in \mathbb{R}^n,\ p, q \text{ as above}.$$

3-4. Consider $(\mathbf{X}, \|\cdot\|_{\infty,1})$, where $\mathbf{X} = \mathcal{C}^1[0,1]$ and $\|f\|_{\infty,1} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$ and also consider $(\mathbf{Y}, \|\cdot\|_\infty)$, where $\mathbf{Y} = \mathcal{C}[0,1]$.

3-4-1. Show that $T = \dfrac{d}{dx}\colon \mathbf{X} \to \mathbf{Y}$ is a bounded linear operator. [7]

3-4-2. Show that $T = \dfrac{d}{dx}\colon D(T) \subsetneq \mathbf{Y} \to \mathbf{Y}$ is an unbounded linear operator, where $D(T) = \mathcal{C}^1[0,1]$. [10]

(Hint: use $u_n(x) = \sin(n\pi x)$).

God bless you !!!